Rotation Dynamics

What do we gain from the perspective of the points on circle as a set of equivalence classes?

Recall) The circle can be thought of as the set of equivalence classes

$$[x] := \{ \ldots, -4\pi + x, -2\pi + x, x, x + 2\pi, \ldots \}$$

"such that" the integers

$$:= \{ k 2\pi + k : k \in \mathbb{Z} \}$$

need to tell what the conditions on K are

in words:

"the set of all numbers $2\pi k + x$, where K is an integer" yesterday: the geometry of equiv. classes

today: the $\underbrace{\text{algebra of equivalence classes.}}$

Idea: Develop addition on equivalence classes

Define: $A+B := \{x+y : x \in A, y \in B\}$

example:

$A = \{1,2\}$, $B = \{3,4\}$

$$A+B = \{\overset{1+3}{4}, \overset{1+4}{5}, \overset{2+3}{5}, \overset{2+4}{6}\}$$

$= \{4,5,6\}$ ← duplicates listed once

Theorem) For any equiv. classes $[x]$, $[y]$, $[x]+[y] = [x+y]$.

Proof) We'll show that $[x]+[y] \subseteq [x+y]$ and that $[x+y] \subseteq [x]+[y]$.

First: choose some $z \in [x]+[y]$.

By definition of set addition,

$$z = (x + k \cdot 2\pi) + (y + \ell \cdot 2\pi)$$

for some $k, \ell \in \mathbb{Z}$.

factor out 2π

$$= (x+y) + 2\pi(k+\ell)$$

Since $k+\ell$ is an integer, $z \in [x+y]$.

Now, we show that $[x+y] \subset [x]+[y]$.

Let $w \in [x+y]$.

So by definition

$$w = x + y + 2\pi m$$

can split m into the sum of any two integers whose sum is m. Simplest choice is 0 and m,

$$w = (x) + (y + 2\pi m)$$

Since $x \in [x]$ and $y + 2\pi m \in [y]$, we have that $w \in [x]+[y]$.

$\square$

means we've proved the thing we wanted to prove.

Observations

identity property

(a) $[0]+[x]=[x]$

associativity property

(b) $([x]+[y])+[z] = [x]+([y]+[x])$

existance of inverses

(c) $[-x]+[x] = [0]$

these properties give that $\mathbb{R}/2\pi\mathbb{Z}$ is a group with the operation of set addition

bonus property

(d) $[x]+[y] = [y]+[x]$

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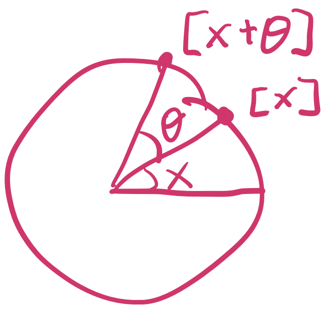


Define $R_\theta : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$

$$R_\theta([x]) = [x+\theta] = [x] + [\theta]$$

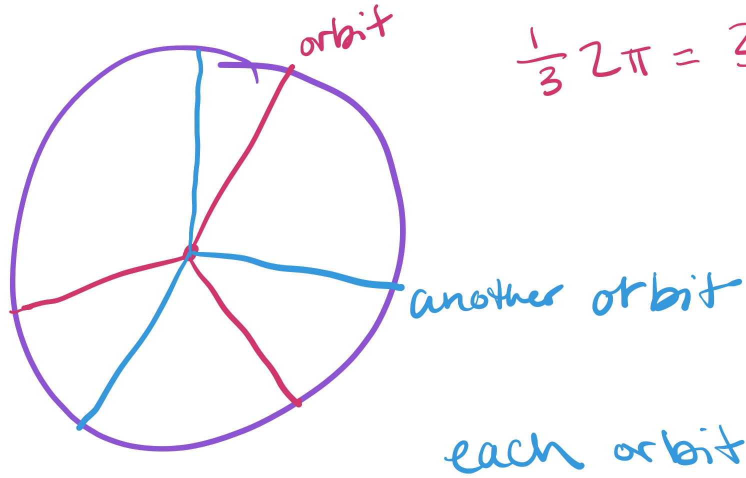
This is exactly the circle rotation by angle θ.

Goals ↙ "Theorem"

Note: these do not imply one another

Thm If $\theta = \frac{P}{Q} \cdot 2\pi$, $P, Q \in \mathbb{Z}$, every point on the circle has period Q.

Thm If there exists a periodic point of period $q \in \mathbb{Z}$ for the circle rotation by θ, then $\theta = 2\pi \frac{p}{q}$ for some $p \in \mathbb{Z}$.



$\frac{1}{3} 2\pi = \frac{2\pi}{3}$

each orbit has 3 points and are disjoint

A formula for $R_\theta^k([x])$

$R_\theta^k([x]) = [x + k\theta]$ for all ~~$k \in \mathbb{Z}$~~ $k \geq 1$.

<u>Proof</u>) (by induction)

$k=1 \Rightarrow k=2 \quad k=3 \quad \ldots \quad k=99 \quad k=100 \quad \ldots$

write a proof for base case

then prove if I know previous one, I can deduce the next one
the inductive step

works like dominos

__Proof of base case)__

Check that $R_\theta^1([x]) = [x + 1\theta]$

true from the definition

__Proof of inductive step)__ Assume that it holds for k and prove that it holds for $k+1$.

Assume it holds for $R_\theta^k([x]) = [x + k\theta]$

Then $R_\theta^{k+1}([x]) \overset{\text{by def of iterates}}{=} R_\theta(R_\theta^k([x]))$

by inductive assump

$= R_\theta([x + k\theta])$

def of R_θ

$= [x + k\theta] + [\theta]$

property of set addition

$= [x + k\theta + \theta]$

factoring

$= [x + \theta(k+1)]$

$\square$